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ON SOME DYNAMIC LINEAR PROGRAMMING PROBLEMS

Richard Bellman

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ON SOME DYNAMIC LINEAR PROGRAMMING PROBLEMS

Richard Bellman

Summary

A study is made of a class of mathematical problems connected with physical situations which require that a finite or unbounded sequence of operations be performed for the purpose of achieving a desired result. The only case considered here is that in which each operation performs a mapping of the parametric space onto itself. ( )

§1. Introduction.

We are interested in a class of mathematical problems connected with physical situations which require that a bounded or unbounded sequence of operations be performed for the purpose of achieving a desired result. The result of each individual operation is a stochastic event which yields information to be used in planning the subsequent operations. We consider here only the case where each operation, performed upon a system whose state is specified by a set of parameters, has the effect of converting one state into another. This last is equivalent to saying that each operation performs a mapping of the parameter space onto itself.

The two principal problems encountered in situations of this type are those of maximizing the yield obtained in a given time, or of minimizing the time required to accomplish a certain task. Since we are dealing with a sequence of stochastic events we shall have to employ the metrics of probability theory, and the two types of criteria we will use are:

1. The expected yield, or time.
2. The probability of achieving a certain goal, i.e., the probability that the yield is greater than a given quantity, or that the time is less than a given time.

We shall see below that the choice of an appropriate criterion plays a decisive rôle in determining the form of the solution to the problems we consider.

An extensive utilization in the theory of statistical inference of the fundamental idea that the results of the preceding operations should be used to guide the course of the subsequent operations is due to Wald in his theory of sequential analysis.

However, the full range of problems of this type does not seem to have been appreciated, nor does the class of mathematical problem encountered seem to have been discussed at any length. The purpose of the present paper is to indicate some of the vast number of problems which may be treated from the above viewpoint and to indicate briefly the nature of the solutions. A more complete exposition will be given elsewhere.

Some of the problems we discuss here are related to those discussed by Arrow, Blackwell and Girshick,[(1)] and Wald and Wolfowitz,[(2)].

All of these problems could be included as particular cases of an abstract formulation. However, it is important to postpone this until a large number of individual problems have been solved, since certain indigenous features of each problem will facilitate its solution.

What ties the variety of problems we consider together is a point of view which affords an immediate foothold. In order to present this concept we must first discuss what is meant by a solution of a problem of this type. By a solution we mean a set of rules which tell us which operation to perform at every stage in every situation. Since the number of possibilities in some of the problems involving unbounded sequences is quite large, a listing of all possible optimal sequences is neither feasible nor elegant nor useful.

To avoid these unattractive features we begin by observing that we have stipulated that the result of each operation is to change the system into a similar system determined by different parameters. In light of this, the question arises as to whether or not it might not be sufficient to prescribe merely the best first move in each situation that arises. We are immediately, however, confronted by a difficulty. Our original purpose was to optimize

according to a certain criterion, of the type given above. It may not be true that after the first stage, the optimal continuation is to be made according to the same criterion.

We are led naturally to direct our attention to those criteria which possess a certain invariant property, namely, after any finite number of operations, the optimal continuation is obtained by employing the original criterion to guide the subsequent operations.

Some examples of criteria having this property are:

1. Maximize the expected yield, or minimize the expected time.
2. Maximize the probability of success.

Instances of criteria not possessing this property are:

1. Maximize the expected yield at the end of N stages.
2. Maximize the probability of success in N or fewer stages.

Let us now turn to the discussion of some specific problems.

§2. Determination of the state of a physical system.

Frequently it is necessary to determine by testing the parameters of state of a physical system where it is not possible to examine the entire parameter space at one time. Since each test requires time, the question arises naturally of arranging the testing program to consume the least time. In this case, we may use either the criterion of minimizing expected time, or the probability of determining the state in time less than T. It is clear that we would test for the most likely state first. We make the problem more interesting if we attach to each state a probability that the testing device will not register, or

what is more drastic, that it will be destroyed before registering any information.

We may also consider the situation where each test disturbs the system. These problems are more difficult than those above.

As a simple model of a problem of the first type, consider the problem of finding a ball which is known to be in one of N boxes, with a probability distribution p_k that it is in the k^{th} box. Let q_k be the probability, independent of previous trials, that we are unable to look into the k^{th} box when we examine it and assume for simplicity that each examination consumes one unit of time. We wish to determine the testing procedure which minimizes the expected time required to find the ball.

Before proceeding, let us make precise what we mean by finding the ball. We may merely wish to locate the box containing it or we may wish actually to obtain or view the ball itself. It is clear that for a sizable number of boxes there will be little difference between the expected times required to accomplish either purpose.

If we use the criterion of expected time to obtain or view the ball, the solution is extraordinarily simple: at each stage, examine the box for which $p_k(1 - q_k)$ is largest. Here (p_k) is the probability distribution before the N^{th} test.

Let us sketch the method of proof. Let $E_N(p_1, p_2, \cdots, p_N)$ be the expected time required using the optimal procedure. Then an enumeration of possibilities yields the functional equation

$$(1) \qquad E_N(p_1, p_2, \cdots, p_N) = \min_{\ell} \Bigg[q_\ell[1 + E_N(p_1, p_2, \cdots, p_N)] + (1 - q_\ell)\Bigg[p_\ell + (1 - p_\ell)\Bigg[1 + E_{N-1}\Bigg(\frac{p_1}{1 - p_\ell}, \cdots, \frac{p_N}{1 - p_\ell}\Bigg)\Bigg]\Bigg]\Bigg]$$

where the term corresponding to p_ℓ is omitted in E_{N-1}. Having established by

direct calculation the result for $N = 2$, it is easy to proceed inductively and obtain the result stated above.

If we use the alternative criterion, that of merely locating the ball, certainly a more natural one, we obtain the same functional equation. However, the case $N = 2$ is now quite different due to the fact that having looked in either box and not having found it immediately locates the ball in the other box. Hence for $N = 2$ one would examine the box for which q_1 is largest, unless $p_1 = 0$ or 1. There is an interesting discontinuity here which persists throughout. The optimal procedure for $p_1 \neq 0, 1$ does not furnish as $p_1 \rightarrow 0$ or 1 the optimal procedures for these cases. We will find in the case of N boxes that the $(p_1, p_2, \cdots, p_N)$ space is divided into N regions, R_i, $i = 1, \cdots, N$, having the property that if $P(p_1, p_2, \cdots, p_N) \in R_k$, then the k^{th} box is to be examined first. These regions may be computed readily, beginning with $N = 3$, using the functional equation (1).

This functional equation seems to be of a type not previously encountered.

In view of the above result, we see that the choice of an appropriate criterion is of great importance. It is conceivable that in other problems of sequential analysis a slight change in the criterion may materially simplify the mathematical treatment.

Results similar to the above hold if we introduce probabilities of breakage of the instrument and consider as a criterion the probability of finding or obtaining the ball.

§3. Testing for information.

Suppose that we own one testing device and have a number of objects to be tested, each of which possesses a certain amount of information. We may assume

that there is no possibility of breakage of the instrument and ask for the testing procedure which yields the maximum information in a given time, or we may assume that the instrument is breakable and ask for the testing policy which yields the maximum information before the instrument is destroyed. In the first case assume further that there is a set of probability distributions which govern the information obtained from testing the k^{th} object. It can be shown inductively via the functional equation, or using the fact that the order of testing is immaterial in this case once one has decided to test the k^{th} object a fixed number of times, that an optimal procedure is one which maximizes the expected information obtained at each step,[3].

In the second case where we attach to each object a probability q_k that the instrument is destroyed in the course of testing, it is no longer true that one maximizes the expected information obtained at each step. Rather, one maximizes the E_k/q_k where E_k is the expected information obtained from testing the k^{th} object.

Since the form of the solution is independent of N (which is not obvious beforehand), it is sufficient to determine the solution for $N = 2$. Here an intuitive method rapidly leads to the solution. The only parameters which change from test to test are the amounts of information contained in each object, which we call I , II . We suspect that the (v_1, v_2) positive quadrant will be divided into two regions, R_1, R_2, with the property that if $(v_1, v_2) \in R_i$ the i^{th} object is tested on the first step. These regions will be determined if we can find the common boundary, which is the locusof points (v_1, v_2) for which it is a matter of indifference as to which object is tested first. If we write down the functional equation corresponding to (1) of §2 and equate the right-hand sides, we find that we obtain little information since the unknown function itself appears.

One step further, however, yields the solution. The boundary curve C separating R_1 and R_2 should have the property that if $(v_1, v_2) \in C$ and the first object is tested, then the new (v_1, v_2) point is in R_2, and conversely. Hence the symbolic equation (I) = (II) leads to (I · II) = (II · I). When this equation is written out the unknown functions on both sides cancel and we are left with a linear equation in v_1 and v_2 which is precisely $E_1/q_1 = E_2/q_2$, [4].

The problem becomes more interesting and decidedly more difficult if we allow the possibility of one instrument testing several objects in one operation or consider more testing devices being used simultaneously. A consideration of the simplest such case shows very clearly the difficulties that are encountered in attempting a general theory.

Consider two objects, I and II, and a testing device which may be used to test either one separately or both together. We consider three possibilities only, no information, complete information, or a fixed percentage of the information obtained. Let

(1) $\qquad p_i$ = probability instrument is not destroyed testing i^{th} object,

and if not destroyed,

$\qquad r_i$ = probability of complete information,

$\qquad s_i$ = probability of no information

$\qquad t_i$ = probability of σ of the information obtained.

Let $f(v_1, v_2)$ = expected information obtained using optimal policy. Then

(2)
$$f(v_1, v_2) = \max_{1,2}\left\{\left[p_1\left[r_1\left[v_1 + f(0, v_2)\right] + s_1\left[f(v_1, v_2)\right] + t_1\left[\sigma v_1 + f\left((1 - \sigma)v_1, v_2\right)\right]\right]\right], [1 \longrightarrow 2]\right\},$$

where $[1 \longrightarrow 2]$ indicates the similar expression obtained from testing II first.

We would expect, from what has preceded, that there will be three regions of the positive quadrant R_1, R_2, R_{12} with the properties that if $(v_1, v_2) \in R_1$, R_2 or R_{12}, one tests I, II, or both. If we attempt to use the previous arguments to determine the boundary, we find that the method fails, since a point on a boundary can move into either of two regions and it is not simple to determine which.

To overcome this difficulty, we use a continuity method, taking σ as a parameter which varies between 0 and 1. At each extreme it is easy to determine the three regions. As σ varies we may follow the boundaries and show that for all σ these three regions exist. The boundaries will depend upon the probabilities listed in (1).

Notice that in the above we have taken the σ's to be equal in all three operations. It would be desirable to take them as distinct. We can use one σ for tests of I and II and another for simultaneous tests of I and II and the method applies equally. However, if all the σ's are distinct, not only does our method fail, but there are examples showing that the method fails,[5].

The essential fact that makes this problem tractable is a periodicity in the tests that occurs in the special case, but not the general case of distinct σ's.

§4. Other problems.

Let us in this section merely mention some other problems of this general class.

An Investment Problem:

Suppose a person is left a sum of money to live on for the rest of his life. He would like to use all the money before he dies, but he does not wish to be left penniless while alive. How should he apportion his money between

consumption and investment if he wishes to consume the maximum (expected) amount of money before he dies?

Problems of this type are also encountered in a treatment of optimal gambling methods, or in the theory of games in general if one assumes that one at least of the players has a finite amount of money.

A Logistics Problem:

Suppose that we have n ports, A_1, A_2, $\cdots$, A_n, and let the i^{th} port have a_i ships available at the initial time, and a_{ij} shiploads to be sent to the j^{th} port, $a_{ii} = 0$. How does one route the ships so as to minimize the total transit time?

For the case where equal times are consumed sailing between any two ports, a very elegant solution has been given by E. W. Paxson, depending only upon conservation principles.

Finally, we note that there are many problems in organization theory relating to the communication of information and the performance of many-person tasks which may be considered to be problems of the general type considered here.

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(1) KENNETH J. ARROW, D. BLACKWELL and MEYER A. GIRSHICK, "Bayes and Minimax Solutions of Sequential Decision Problems," Econometrica, Vol. 17, 1949, pp. 213-244.

(2) A. WALD and J. WOLFOWITZ, "Optimum Character of the Sequential Probability Ratio Test," Annals of Mathematical Statistics, Vol. 19, 1948, pp. 326-339.

(3) This result was first obtained by D. Blackwell using a different technique.

(4) This result was obtained in another way by M. Shiffman and the author.

(5) Private communication of H. N. Shapiro and S. Karlin.